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# INVESTIGATION OF THE HIGH ENERGY BEHAVIOURS OF THE SCALAR PARTICLE SCATTERING AMPLITUDE IN THE GRAVITATIONAL FIELD BY FUNCTIONAL APPROACH[†]

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**Abstract**

Closed expressions for the Green function and amplitude of the scalar particle scattering in the external gravitational field $g_{\mu\nu}(x)$ are found in the form of functional integrals. It is shown that, as compared with the scattering on the vector potential, the tensor character of the gravitational field leads to a more rapid increase of the cross section with increasing energy. Discrete energy levels of particles are obtained in the Newton potential.

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# 1 Introduction

The method of solving equations for the one-particle Green function in the external field by the functional integral has been given by Barbashov [1]. A closed solution of the particle Green function is effective in studying particle interactions in the infrared and high energy regions [2-4]. In this note, we present a generalization of this method to nonlinear interactions including the derivative coupling. We consider a scalar field $\Psi(x)$ interacting with a gravitational field $g_{\mu\nu}$. The closed solution of the scalar particle Green function in the external field $g_{\mu\nu}$ has obtained. To contrast with the work [5] where the problem has been considered an additional condition on the field $g_{\mu\nu}$ /harmonic condition/ is explicitly taken into account in our method. The scalar particle Green function represented by the functional integral is used for constructing the scattering amplitude. In the high energy region the eikonal representation for the scattering amplitude of the scalar particle in the tensor potential is obtained. The Newton potential is considered as an example. It is shown that as compared with the scattering on vector potential, the character of the gravitational field leads to a more rapid increase in the cross section with increasing energy. Discrete energy levels of particles are obtained in the Newton potential.

# 2 The scalar particle Green function in an external gravitational field

Let us consider the model of interaction of a scalar field with a gravitational field $g_{\mu\nu}$ where the interaction Lagrangian is of the form

$$L(x) = \frac{\sqrt{-g}}{2}\left[g^{\mu\nu}(x)\partial_\mu\Psi(x)\partial_\nu\Psi(x) - m^2\Psi(x)^2\right] \ , \tag{1}$$

where

$$g = det g_{\mu\nu} = \det\sqrt{-g}g^{\mu\nu}(x) \ .$$

Variating the Lagrangian (1) leads to the following equation for field $\Psi(x)$:

$$\left[-\tilde{g}^{\mu\nu}(x)\partial_\mu\partial_\nu - \sqrt{-g}m^2 - \partial_\mu\tilde{g}^{\mu\nu}(x)\partial_\nu\right]\Psi(x) = 0 \ , \tag{2}$$

$$\tilde{g}^{\mu\nu}(x) = \sqrt{-g}g^{\mu\nu}(x) \ .$$

Equation (2) is conveniently investigated in the harmonic coordinates defined by the condition

$$\partial_\mu\tilde{g}^{\mu\nu}(x) = 0 \ . \tag{3}$$

The harmonic gauge (3) being the analogy of the Lorentz gauge in the electrodynamics has led to eliminate the nonphysical component of the tensor field. Taking Eq.(3) into account, Eq.(2) yields to:

$$\left[\tilde{g}^{\mu\nu}(x)i\partial_\mu i\partial_\nu - \sqrt{-g}m^2\right]\Psi(x) = 0 \ .$$

For the scalar particle Green function in the gravitational field we have the following equation

$$\left[\tilde{g}^{\mu\nu}(x)i\partial_\mu i\partial_\nu - \sqrt{-g}m^2\right] G(x,y|g^{\mu\nu}) = -\delta^{(4)}(x-y) \ . \tag{4}$$

Equation (4) can be written in an operator form, if one uses the representation of the inverse operator $[g^{\mu\nu}(x)i\partial_\mu i\partial_\nu - \sqrt{-g}m^2]^{-1}$ proposed by Fock and Feynman [7, 8] as an exponent form

$$G(x,y|g^{\mu\nu}) =$$
$$i\int_0^\infty d\tau \exp\left(-im^2\int_0^\tau \sqrt{-g(x,\xi)}d\xi + i\int_0^\tau \tilde{g}^{\mu\nu}(x,\xi)i\partial_\mu(\xi)i\partial_\nu(\xi)d\xi\right)\delta^{(4)}(x-y) \ . \tag{5}$$

In this notation an exponent, whose the coefficient has noncommuting quantities as $\partial_\mu(x)$, $\tilde{g}^{\mu\nu}$ and $g(x)$, is considered as $T_\xi$-exponent, where $\xi$ plays the role of an ordering index. The coefficient of the exponent in the Eq.(5) is quadratic in the differentiation operator $\partial_\mu$. However, the transition from $T_\xi$-exponent to an ordinary operator expression /"disentangling" the operators by the terminology of Feynman/ can not be performed without the series expansion. But one can lower the power of the operator $\partial_\mu$ in Eq.(5) if one uses the following formal transformation [1]:

$$\exp\left(i\int_0^\tau d\xi \tilde{g}^{\mu\nu}(x,\xi)i\partial_\mu(\xi)i\partial_\nu(\xi)\right) =$$
$$C_\nu \int \prod_\eta d^4\nu(\eta) exp\left(i\int_0^\tau [\tilde{g}^{\mu\nu}(x,\xi)]^{-1}\partial_\nu(\xi)\partial_\nu(\xi) - 2i\int_0^\tau d\xi \nu^\mu(\xi)\partial_\mu(\xi)\right) \ . \tag{6}$$

The functional integral in the right-hand side of the Eq.(6) is taken in the space of 4-dimensional functions $\nu_\mu(\xi)$. The constant $C_\nu$ is defined by the condition

$$C_\nu \int \delta^4\nu_\mu exp\left(-i\int_0^\tau d\xi[\tilde{g}^{\mu\nu}(x,\xi)]^{-1}\nu_\mu(\xi)\nu_\nu(\xi)\right) = 1 \ ,$$

from which it follows

$$C_\nu = \left[\int \delta^4\nu_\mu exp\left(-i\int_0^\tau d\xi[\tilde{g}^{\mu\nu}(x,\xi)]\right)^{-1}\nu_\mu(\xi)\nu_\nu(\xi)]\right]^{-1} = (det[\tilde{g}^{\mu\nu}(x,\xi)])^{-1/2} \ .$$

After substituting (6) into (5), the operator $exp\,(-2\int_0^\tau \nu^\mu(\xi)\partial_\mu(\xi)d\xi)$ can be "disentangled" and we can find solution of the equation (4) in the form of the functional integral

$$G(x,y|g^{\mu\nu}) = -i\int_0^\infty d\tau e^{-im^2\tau} \cdot$$
$$\cdot\, C_\nu \int \delta^4\nu exp\left(-im^2\int_0^\tau [\sqrt{-g(x_\xi)}-1]d\xi - i\int_0^\tau d\xi[\tilde{g}^{\mu\nu}(x_\xi)]^{-1}\nu_\mu(\xi)\nu_\mu(\xi)\right) \cdot$$
$$\cdot\, \delta^{(4)}\left(x-y-2\int_0^\tau \nu(\eta)d\eta\right), \tag{7}$$

where

$$x_\xi = x - 2\int_\xi^\tau \nu(\eta)d\eta \ .$$

The Fourier transform of the Green function $G(x, y/g^{\mu\nu})$ has the following form

$$G(p, q|g^{\mu\nu}) = \int dxdy e^{ipx-iqy} G(x, y/g^{\mu\nu}) =$$

$$i\int_0^\infty e^{i(p^2-m^2)\tau} \int dy e^{i(p-q)y} \cdot$$

$$\cdot C_\nu \int \delta^4\nu exp\left(-im^2\int_0^\tau [\sqrt{-g(y_\xi)}-1]d\xi(-i)\int_0^\tau [\tilde{g}^{\mu\nu}(y_\xi)]^{-1}[\nu(\xi)+p]_\mu[\nu(\xi)+p]_\nu\right) , \quad (8)$$

$y_\xi = y + \int_0^\xi [\nu(\eta)+p]d\eta$ .

Eq.(8) is the closed expression for the scalar particle Green function in the external gravitational field. Let us now consider the gravitational field in the linear approximation, i. e put $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $\eta_{\mu\nu}$ is the Minkowski metric tensor $\eta_{\mu\nu} = diag(1, -1, -1, -1)$. Rewrite Eq.(8) in the variables $h_{\mu\nu}(x)$ after dropping out the terms with an exponent power higher than first $h_{\mu\nu}(x)$*

$$G(p, q|h^{\mu\nu}) =$$

$$i\int_0^\infty d\tau e^{i(p^2-m^2)} \int dy e^{i(p-q)y} \int \delta^4\nu exp\left(-i\int_0^\tau \nu_\mu(\xi)\nu^\mu(\xi)d\xi + i\int_0^\tau M(y_\xi)d\xi\right) , \quad (9)$$

where

$$M(y_\xi) = \left(\frac{m^2}{2}h_\nu^\nu(y_\xi) + [\nu(\xi)+p]_\mu[\nu(\xi)+p]_\nu h^{\mu\nu}(y_\xi)\right) . \quad (10)$$

# 3 Eikonal representation for the scattering amplitude of the scalar particle on the tensor potential

In this section we consider the scattering of the scalar particle in the external potential $h_{\mu\nu}$. Using expression (9) we find the scattering amplitude $F(p, q|h_{\mu\nu})$ / $p$ and $q$ are the particle momenta before and after the scattering respectively / by the following formula:

$$F(p, q|h^{\mu\nu}) = \lim_{p^2,q^2\to m^2} (p^2-m^2)(q^2-m^2)G(p, q|h^{\mu\nu}) . \quad (11)$$

Taking into account the identity $e^a - 1 = a\int_0^1 e^{\lambda a}d\lambda$, we subtract from $G(p, q|h^{\mu\nu})$the free Green function $G_0(p, q) = (2\pi)^4\delta^4(p-q)/p^2-q^2$ which does not give contribution to the scattering amplitude (11). As result, we obtain

$$F(p, q|h^{\mu\nu}) = \lim_{p^2,q^2\to m^2} (p^2-m^2)(q^2-m^2)i\int_0^\infty e^{iTy}\int d\tau e^{i(p^2-m^2)\tau} \cdot$$

*Lagrangian (1) in the linear approximation to $h^{\mu\nu}$ has the form $L(x) = L_0 + L_{int}$ , where

$$L_0 = \frac{1}{2}[\partial^\mu\Psi(x)\partial_\mu\Psi(x) - m^2\Psi^2(x)] ,$$

$$L_{int}(x) = -\frac{1}{2}h^{\mu\nu}(x)T_{\mu\nu},$$

$$T_{\mu\nu}(x) = \partial_\mu\Psi(x)\partial_\nu\Psi(x) - \frac{1}{2}\eta_{\mu\nu}[\partial^\mu\Psi(x)\partial_\mu\Psi(x) - m^2\Psi^2] ,$$

$T_{\mu\nu}(x)$-the energy momentum tensor of the scalar field.

$$\cdot \int \delta^4 \nu exp[-i \int_0^\tau \nu_\mu(\xi)\nu^\mu(\xi)d\xi] \int_0^\tau d\tau M(y_\xi) \int_0^1 d\lambda e^{i\lambda \int_0^\tau d\xi M(y_\xi)} \ , \tag{12}$$

where $T = p - q$.

Changing in Eq.(12) the variables

$$y = y' - 2p\eta - 2\int_0^\eta \nu(\eta')d\eta' \ ,$$
$$\nu(\xi) = \nu'(\xi) - (p-q)\theta(\eta - \xi) \ , \tag{13}$$

$$\theta = \begin{cases} 1 & \xi > 0, \\ 0 & \xi < 0, \end{cases}$$

and using the relation [9]

$$\lim_{a,\epsilon \to 0} ia \int_0^\infty d\tau e^{ia\tau - \epsilon\tau} f(\tau) = f(\infty) \ , \tag{14}$$

we pass to the mass shell. Finally the scattering amplitude has the form

$$F(p, q|h^{\mu\nu}) =$$

$$\int dy e^{iTy} \int \delta^4 \nu exp(-i \int_{-\infty}^\infty \nu_\mu(\xi)\nu^\mu(\xi)d\xi) M(y/0) \int_0^1 d\lambda exp[i\lambda \int_{-\infty}^\infty d\xi M(y/\xi)], \tag{15}$$

where

$$M(y|\xi) = \frac{m^2}{2} h^\mu_\nu [y + 2p\theta(\xi)\xi + 2q\theta(-\xi)\xi + 2\int_0^\xi \nu(\eta)d\eta] +$$
$$[\nu(\xi) + p\theta(\xi) + q\theta(-\xi)]_\mu [\nu(\xi) + p\theta(\xi) + q\theta(-\xi)]_\nu \ \cdot$$
$$\cdot \ h^{\mu\nu}[y + 2p\theta(\xi) + 2q\theta(-\xi)\xi + 2\int_0^\xi \nu(\eta)d\eta] \ . \tag{16}$$

For the calculation of the functional integrals we use the straight line path approximation [3, 4, 10], i. e. we assume that in the high energy particle scattering on the smooth potential $h^{\mu\nu}$, one can be neglect the dependence on the functional variables $\nu_\mu(\eta)$. In other words, it is considered that the main contribution to the functional integral (15) comes from a trajectory particle moving freely from the momentum $\vec{p}$ with $\xi > 0$ to the momentum $\vec{q}$ with $\xi < 0$ and passing via the point $y$ with $\xi = 0$.

To simplify, we consider the case when the potential does not depend on the time:$h^{\mu\nu}(y) = h^{\mu\nu}(\vec{r}, t) = h^{\mu\nu}$. Therefore, for the scattering amplitude we obtain the following closed expression†

$$F(p, q) = 2(2\pi)^2 \delta(p_0 - q_0) f(p, q),$$

† the amplitude $f(p, q)$ is normalized by the relations

$$\sigma = \frac{4\pi}{|\vec{p}|} Im f(p, p), \frac{d\sigma}{d\Omega} = |f(p, q)|^2.$$

$$f(p,q)=\frac{1}{4\pi}\int d\vec{r}e^{iT\vec{r}}M'(\vec{r}/0)\int_0^1 d\lambda exp[i\lambda\chi(\vec{r})], \tag{17}$$

where

$$M'(\vec{r}|0)=\left(\frac{m^2}{2}h_\nu^\nu(\vec{r})+\frac{1}{4}[p+q]_\mu[p+q]_\nu h^{\mu\nu}(\vec{r})\right) \ ,$$

$$\chi_0(\vec{r})=\frac{1}{2\mid\vec{p}\mid}\int_{-\infty}^{\infty}ds[(\frac{m^2}{2}h_\nu^\nu(\vec{r}+\hat{p}\theta(s)s+\hat{q}\theta(-s)s)]+$$

$$\frac{1}{2\mid\vec{p}\mid}\int_{-\infty}^{\infty}ds\left([p\theta(s)+q\theta(-s)]_\mu[p\theta(s)+q\theta(-s)]_\nu h^{\mu\nu}(\vec{r}+p\theta(s)s+\hat{q}\theta(-s)s)\right) \ , \tag{18}$$

$\hat{p}=\frac{|\vec{p}|}{|p|}$. In the case of weak gravitational field, the tensor $h^{\mu\nu}$ has the form [11]

$$\begin{gathered} h^{00}(\vec{r})=2\phi(\vec{r}) \ , \\ h^{\alpha\beta}(\vec{r})=2\delta_{\alpha\beta}\phi(\vec{r}) \ , \\ h^{0\alpha}(\vec{r})=h^{\alpha0}(\vec{r})=0 \ , \\ (\alpha,\beta=1,2,3). \end{gathered} \tag{19}$$

Since we consider the high energy scattering, the first term in Eq.(18) for $\chi_0(\vec{r})$ can be neglected in comparison with the second one. As a result, we have

$$f(p,q)=\frac{1}{2\pi}\int d\vec{r}e^{iT\vec{r}}(p_0^2+\vec{p}^2)\phi(\vec{r})\int_0^1 d\lambda exp(i\lambda\chi_0(\vec{r})) \ , \tag{20}$$

where

$$\chi_0(\vec{r})=\frac{1}{\mid\vec{p}\mid}\int_{-\infty}^{\infty}ds(p^2+\vec{p}^2)\phi(\vec{r}+\hat{p}s) \ . \tag{21}$$

We direct the axis $z$ along the momentum $\vec{p}$, but we place the $x$ axis into the plane defined by the vectors $\vec{p}$ and $\vec{q}$. In this coordinate system, for the phase (21) in the presence of small angles $\theta\ll 1$ we obtain the following expression:

$$\chi_0(p_0,\vec{r})=\frac{2p_0^2}{\mid\vec{p}_z\mid}\left(\int_0^\infty ds\phi(x,y,z+s)+\int_{-\infty}^0 ds\phi(x+s\sin\theta,y,z+s\cos\theta)\right)\cong$$

$$=\frac{2p_0^2}{\mid\vec{p}_z\mid}\int_{-\infty}^{\infty}ds\phi(x,y,z+s)=\frac{2p_0^2}{\mid\vec{p}_z\mid}\int_{-\infty}^{\infty}dz\phi(\vec{r}) \ . \tag{22}$$

At small angles the momentum transfer is nearly perpendicular to the $z$-axis therefore, in Eq.(20) it can put $exp(i\vec{T}\vec{r})=expi(T_xx+T_yy)$. Integrating over $dz$ and $d\lambda$ with taking account of (22), we find the Glauber-type representation for the scattering amplitude [13]

$$f(p,q)=-\frac{i\mid\vec{p}_z\mid}{2\pi}\int d^2\vec{b}e^{i\vec{b}\vec{T}_\perp}\left(\exp[i\chi_0(p_0,\vec{b})]-1\right) \ , \tag{23}$$

where $\vec{b}=(x,y,0)$, and the eikonal phase $\chi_0(p_0,b)$ is determined by Eq.(22). Let us consider the Newton potential as limit of the Yukawa potential when $\mu\to 0$, $\phi(\vec{r})=$

$\left((kMe^{-\mu r})/r\right)_{\mu=0} = (-kM)/r$, where $k$ is the gravitational constant, $k = 6.10^{-39} m_p^2$ and $M$ the mass creating the potential. In this case an eikonal phase (22) get the following form

$$\chi_0(p_0, \vec{b}) = -\frac{2kp_0^2 M}{\mid \vec{p_z} \mid} \int_{-\infty}^{\infty} dz \frac{e^{-\mu\sqrt{b^2+z^2}}}{\sqrt{b^2+z^2}} = -\beta K_0(\mu|b) \ , \tag{24}$$

where $\beta = \frac{kMp_0{}^2}{2\pi|\vec{p}_z|}$, $K_0(\mu|b) = \frac{1}{2\pi}\int d^2\vec{k_\perp} \frac{e^{i\vec{k}_\perp \vec{b}_\perp}}{\vec{k}_\perp^2+\mu^2}$-is the Kelvin function of the zeroth order. For the scattering amplitude, we have the following expression:

$$f(p,q) = -\frac{i|\ \vec{p_z}\ |}{2\pi} \int d^2\vec{b} e^{i\vec{b}\vec{T}} \left( exp[-i\frac{\beta}{2\pi} \int d^2 k_\perp \frac{e^{i\vec{k}_\perp \vec{b}_\perp}}{\vec{k}_\perp^2 + \mu^2}] - 1 \right) \ . \tag{25}$$

Calculate the integral (25), preserving only the terms which does not disappear when $\mu \to 0$

$$f(\vec{p}\vec{q}) = -\frac{kMp_0^2}{\pi t} \frac{\Gamma(1+i\beta)}{\Gamma(1-i\beta)} exp[-i\beta ln(\frac{\sqrt{t}}{\mu c})] \ , \tag{26}$$

where $t = -\vec{T}_\perp^2$; $C$ is the Euler constant. The phase diverging when $\mu \to 0$ in (26) caused, it is well known, by the long-range of action of the Newton potential [14]. From comparing (26) with the result of the work [10], devoted to the consideration of scattering of the scalar particle by the vector potential, one may conclude that $\sigma_{grav.}/\sigma_{vec.} \sim (k^2M^2/e^2)p_0^2$. The poles of the amplitude determined by Eq.(26) give the discrete energy levels of particles in the Newton potential

$$E_n = -\frac{k^2 m^2 M(m+M)}{8\pi^2} \frac{1}{n^2} \ , \tag{27}$$
$$n = 1, 2, 3, ...$$

.

If in the Eq.(27) we put $m = M \sim m_{nuclon}$, we obtain the energy of the ground state equal to

$$E_1 = 9,4.10^{-70} eV \ .$$

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# References

[1] B. M. Barbashov, JETP **48** (1965) 607; E. S. Fradkin, Nucl. Phys. **49** (1963) 624; **76** (1966) 588.

[2] B. M. Barbashov, M. K. Volkov, JETP, **50**(1966) 660.

[3] B. M. Barbashov, D. I. Blokhinshev, V. V. Nesterenko, V. N. Pervushin, Sov. J. Part. Nucl. **v4, No3**(1973) 1.

[4] B. M. Barbashov, S. P. Kuleshov, V. M. Matveev, V. N. Pervushin, A. N. Sissakian, A. N. Tavkhelidze, Phys. Lett. **33B** (1970) 419, 484.

[5] A. Maheshwari, Annals of Physics, **84**(1974) 474.

[6] V. A. Fock, The Theory on Space, Time and Gravity, Fizmatgiz, Moscow, 1961.

[7] V. A. Fock, Phys. Z. Sowietunion, **12**(1937) 474.

[8] R. P. Feynman, Phys. Rev. **84**(1951) 108.

[9] G. A. Milekhin, E. S. Fradkin, JEPT**45**(1963) 1926.

[10] V. N. Pervushin, TMF**4**(1970) 28.

[11] L. P. Landau, E. M. Lifschitz, The Field Theory, "Nauka", Moscow, p. 428, 1973.

[12] R. J. Glauber, Lectures in Theoretical Physics, **v. 1** p. 315, N. Y. 1959.

[13] H.D.I. Abarbanel, C. Itzykson, Phys. Rev. Letters, **23** (1969) 53.

[14] N. F. Mott, H. S. W. Massey, The Theory of Atomic Collisions, ch. 3, Oxeord 1949.